

Witness Rectangle Graphs[★]

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Abstract. In a *witness rectangle graph* (WRG) on vertex point set P with respect to witness point set W in the plane, two points x, y in P are adjacent whenever the open isothetic rectangle with x and y as opposite corners contains at least one point in W . WRGs are representative of a larger family of witness proximity graphs introduced in two previous papers.

We study graph-theoretic properties of WRGs. We prove that any WRG has at most two non-trivial connected components. We bound the diameter of the non-trivial connected components of a WRG in both the one-component and two-component cases. In the latter case, we prove that a graph is representable as a WRG if and only if each component is a connected co-interval graph, thereby providing a complete characterization of WRGs of this type. We also completely characterize trees drawable as WRGs. In addition, we prove that a WRG with no isolated vertices has domination number at most four.

Moreover, we show that any combinatorial graph can be drawn as a WRG using a combination of positive and negative witnesses.

Finally, we conclude with some related results on the number of points required to stab all the rectangles defined by a set of n points.

1 Introduction

Proximity graphs have been widely used in situations in which there is a need of expressing the fact that some objects in a given set—which are assigned to nodes in the graph—are close, adjacent, or neighbors, according to some geometric, physical, or conceptual criteria, which translates to edges being added to the corresponding graph. In the geometric scenario the objects are often points and the goal is to analyze the shape or the structure of the set of spatial data they describe or represent. This situation arises in Computer Vision, Pattern

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Recognition, Geographic Information Systems, and Data Mining, among other fields. The paper [31] is a survey from this viewpoint, and several related papers appear in [44]. In most proximity graphs, given a point set P , the adjacency between two points $p, q \in P$ is decided by checking whether in their *region of influence* there is no other point from P , besides p and q . One may say that the presence of another point is considered an *interference*. There are many variations, depending on the choice of the family of influence regions [16, 31, 34].

Given a combinatorial graph $G = (V, E)$, a *proximity drawing* of G consists of a choice of a point set P in the plane with $|P| = |V|$, for a given criterion of neighborhood for points, such that the corresponding proximity graph on P is isomorphic to G . This question belongs to the subject of *graph drawing problems*, in which the emphasis is on geometrically representing graphs with good readability properties and fulfilling some aesthetic criteria [9]. The main issues are to characterize the graphs that admit a certain kind of representation, and to design efficient algorithms for finding such a drawing, whenever possible.

Proximity drawings have been studied extensively and utilized widely [10, 34]. However, this kind of representation is somehow limited and there have been some attempts to expand the class, for example using weak proximity graphs [11]. Another recently introduced generalization is the concept of *witness proximity graphs* [3, 4], in which the adjacency of points in a given vertex set P is decided by the presence or absence of points from a second point set W —the *witnesses*—in their region of influence. This generalization includes the classic proximity graphs as a particular case, and offers both a stronger tool for neighborhood description and much more flexibility for graph representation purposes.

In the *positive witness* version, there is an adjacency when a witness point is covered by the region of influence. In the *negative witness* version, two points are neighbors when they admit a region of influence free of any witnesses. In both cases the decision is based on the presence or absence of witnesses in the regions of influence, and a combination of both types of witnesses may also be considered; refer to Section 7. Observe that by taking $W = P$ playing a negative role, we recover the original proximity graphs; so this is a proper generalization. Witness graphs were introduced in [3], where the focus is on the generalization of *Delaunay graphs*. The witness version of *Gabriel graphs* was studied in [4], and a thorough exploration of this set of problems is the main topic of the thesis [22].

In this paper, we consider a positive witness proximity graph related to the rectangle-of-influence graph, the witness rectangle graph. The *rectangle of influence graph* $\text{RIG}(P)$, also named the Delaunay graph of a point set in the plane with respect to axis-parallel rectangles [14], is usually studied as one of the basic proximity graphs [34, 35]. In $\text{RIG}(P)$, $x, y \in P$ are adjacent when the *rectangle* $B(x, y)$ they define covers no third point from P ; $B(x, y)$ is the unique open isothetic rectangle with x and y at its opposite corners. In other words, two points are adjacent to each other when they are rectangularly visible [1, 33, 41]. Much effort has been devoted to the rectangle of influence drawability problem [6, 24, 35, 39, 40, 42, 45]. The *witness rectangle graph* (WRG) of vertex point set P (or, simply, *vertices*) with respect to witness point set W

(*witnesses*), denoted $\text{RG}^+(P, W)$, is the graph with the vertex set P , in which two points $x, y \in P$ are adjacent when the rectangle $B(x, y)$ contains at least one point of W . The graph $\text{RG}^+(P, \emptyset)$ has no edges. When W is sufficiently large and appropriately distributed, $\text{RG}^+(P, W)$ is complete. Ichino and Slansky's [30] *mutual neighborhood graph* is precisely the negative-witness version of the witness rectangle graph. See further discussion in Section 6. We also note that a negative-witness version of this graph with $W = P$ would be precisely $\text{RIG}(P)$ discussed above; in fact $\text{RG}^+(P, P)$ is precisely the complement of $\text{RIG}(P)$. An example is shown in Figure 1.

We show in this paper that the connected components of WRGs are geometric examples of graphs with small diameter; these have been attracting attention in the pure graph theory setting [15, 27], and are far from being well understood, even for diameter two [25, 37, 38]. We prove below that the maximum domination number of a WRG with no isolated vertices is four, which we find interesting, considering that the domination number of sufficiently large *planar* graphs of diameter three is at most seven [21, 28].

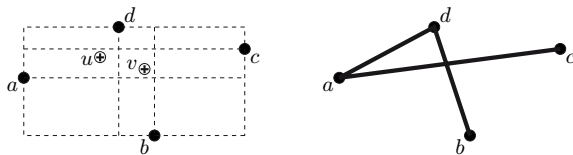


Fig. 1. Left: A set of points $P = \{a, b, c, d\}$ and a witness set $W = \{u, v\}$. Right: the witness rectangle graph $\text{RG}^+(P, W)$. In all our figures for WRGs solid dots denote vertices and dots with a cross denote (positive) witnesses.

Besides some computational issues, such as the construction of $\text{RG}^+(P, W)$ for given sets P and W in an output-sensitive manner, we study several graph-theoretic properties of WRGs: (a) We completely characterize trees drawable as WRGs (Theorem 2). (b) We argue that any WRG has zero, one, or two *non-trivial* connected components (see the definition below and Theorem 3). (c) We prove that the diameter of a single-component WRG is at most six, and that this bound is tight in the worst case (Theorem 3 and subsequent discussion). (d) We prove that the diameter of a (non-trivial) connected component of a two-component WRG is at most three and this can be achieved in the worst case (Theorem 3). (e) In the two-component case, we provide a complete characterization of graphs representable as WRGs. Such graphs, disregarding isolated vertices, are precisely disjoint unions of two connected co-interval graphs (Theorem 4). This last result allows us to recognize in linear time if a combinatorial graph with two non-trivial components can be drawn as a WRG, and to construct such a drawing if it exists. (f) We prove that the maximum domination number of a WRG with no isolated vertices is four.

In Section 6, we give a counterexample to a theorem of Ichino and Slansky in [30]. In Section 7, we show that any combinatorial graph can be drawn as

a WRG with positive and negative witnesses, using a quadratic number of witnesses. Finally, in Section 8, we present some related results on stabbing rectangles defined by a set of points with other points. They can be interpreted as questions on “blocking” rectangular influences.

Terminology and notation. Throughout the paper, we will work with finite point sets in the plane, in which no two points lie on the same vertical or the same horizontal line.

Hereafter, for a graph $G = (V, E)$ we write $xy \in E$ or $x \sim y$ to indicate that $x, y \in V$ are adjacent in G , and generally use standard graph terminology as in [13]. When we speak of a *non-trivial connected component* of a graph, we refer to a connected component with at least one edge (and at least two vertices).

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with disjoint vertex sets, their *join* is the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$ [46].

How to compute a witness rectangle graph. De Berg, Carlsson, and Overmars [12] generalized the notion of dominance, in a way that is closely related to witness rectangle graphs, by defining dominance pairs p, q of a set of points P , with respect to a set O of so-called obstacle points. More precisely, p is said to *dominate* q with respect to O if there is no point $o \in O$ such that p dominates o and o dominates q . Recall that p *dominates* q if and only if $x(p) \geq x(q)$, $y(p) \geq y(q)$, and $p \neq q$.

They prove the following theorem:

Theorem 1 (De Berg, Carlsson, and Overmars [12]). *All dominance pairs in a set of points P with respect to a set of points O can be computed in time $O(n \log n + k)$, where $n = |P| + |O|$ and k is the number of answers.*

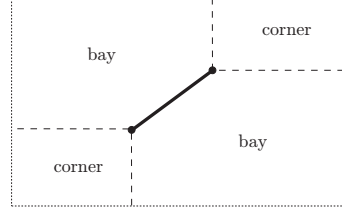
Collecting all dominance pairs in a set of points P with respect to a set of points W , and repeating the procedure after rotating the plane by 90° , one obtains the negative version of the witness rectangle graph. A simple modification of their algorithm yields the positive version:

Corollary 1. *Let P and W be two point sets in the plane. The witness rectangle graph $\text{RG}^+(P, W)$ with k edges can be computed in $O(n \log n + k)$ time, where $n := \max\{|P|, |W|\}$.*

2 Structure of Witness Rectangle Graphs

Let $G := \text{RG}^+(P, W)$ be the witness rectangle graph of vertex set P with respect to witness set W . We assume that the set of witnesses is *minimal*, in the sense that removing any witness from W changes G . Put $n := \max\{|P|, |W|\}$ and let $E := E(G)$ be the edge set of G . We partition E into E^+ and E^- according to the slope sign of the edges when drawn as segments. Slightly abusing the terminology we refer to two edges of E^+ (or two edges of E^-) as having *the same slope* and an edge of E^+ and an edge of E^- as having *opposite slopes*.

Recall that the open isothetic rectangle (or *box*, for short) defined by two points p and q in the plane is denoted $B(p, q)$; for an edge $e = pq$ we also write $B(e)$ instead of $B(p, q)$. Every edge e , say in E^+ , defines four regions as in the figure on the right, that we call (*open*) *corners* and (*closed*) *bays*.



Observation 1 *Every $x \in P$ inside a corner of an edge e is adjacent to at least one endpoint of e .*

Note that for any P , W , and $P' \subset P$, the graph $\text{RG}^+(P', W)$ is an induced subgraph of $\text{RG}^+(P, W)$, so the class of graphs representable as WRGs is closed under the operation of taking induced subgraphs.

Two edges are *independent* when they share no vertices and the subgraph induced by their endpoints contains no third edge. Below we show that G cannot contain three pairwise independent edges, which imposes severe constraints on the graph structure of G .

Lemma 1. *Two independent edges in E^+ (respectively, E^-) cannot cross or share a witness. The line defined by their witnesses is of negative (respectively, positive) slope.*

Proof. Let the two edges be $ab, cd \in E^+$, with $x(a) < x(b)$ and $x(c) < x(d)$. A common witness would have a and c in its third quadrant and b and d in the first, implying $a \sim d$ and $c \sim b$, a contradiction. If ab and cd cross, assume without loss of generality that $x(a) < x(c)$. Neither c nor d can be inside $B(a, b)$ (because of Observation 1) and hence $B(a, d) \cup B(c, b) \supset B(a, b)$, implying $a \sim d$ or $c \sim b$, a contradiction. Finally, the second part of the statement is a direct consequence of Observation 1. \square

Lemma 2. *Two independent edges with opposite slopes must share a witness.*

Proof. Let $ab \in E^+$ and $cd \in E^-$ be independent. Let w be a witness for ab and let w' be a witness for cd . The points c and d are not in quadrants I or III of w , as otherwise the two edges would not be independent. If w is shared, we are done. Otherwise it cannot be that c lies in quadrant II of w while d lies in quadrant IV, or vice versa. Therefore c and d are either both in quadrant II or both in quadrant IV of w . Assume, without loss of generality, the former is true. The witness w' is not outside of $B(a, b)$, as we would have $c \sim a$ and/or $c \sim b$ (assuming, without loss of generality, that $x(c) < x(d)$) and the edges would not be independent. Therefore w' is in $B(a, b)$, so w' is a shared witness, as claimed. \square

Lemma 3. *There are no three pairwise independent edges in E^+ (or in E^-).*

Proof. Assume that three pairwise independent edges e_1, e_2, e_3 in E^+ are witnessed by w_1, w_2, w_3 , respectively, with $x(w_1) < x(w_2) < x(w_3)$. Then, by

Lemma 1, $y(w_1) > y(w_2) > y(w_3)$. By the same lemma at least one endpoint of e_1 is in the second quadrant of w_2 and at least one endpoint of e_3 is in its fourth quadrant, contradicting their independence. \square

Lemma 4. *A witness rectangle graph does not contain three pairwise independent edges.*

Proof. By Lemma 3, two edges ab and cd of the three pairwise independent edges ab , cd , and ef have opposite slopes. By Lemma 2, ab and cd share a witness point w . Every quadrant of w contains one of the points a , b , c , or d , therefore both e and f must be adjacent to one of them, a contradiction. \square

The preceding results allow a complete characterization of the trees that can be realized as WRGs. An analogous result for rectangle-of-influence graphs was given in [35].

Theorem 2. *A tree is representable as a witness rectangle graph if and only if it has no three independent edges.*

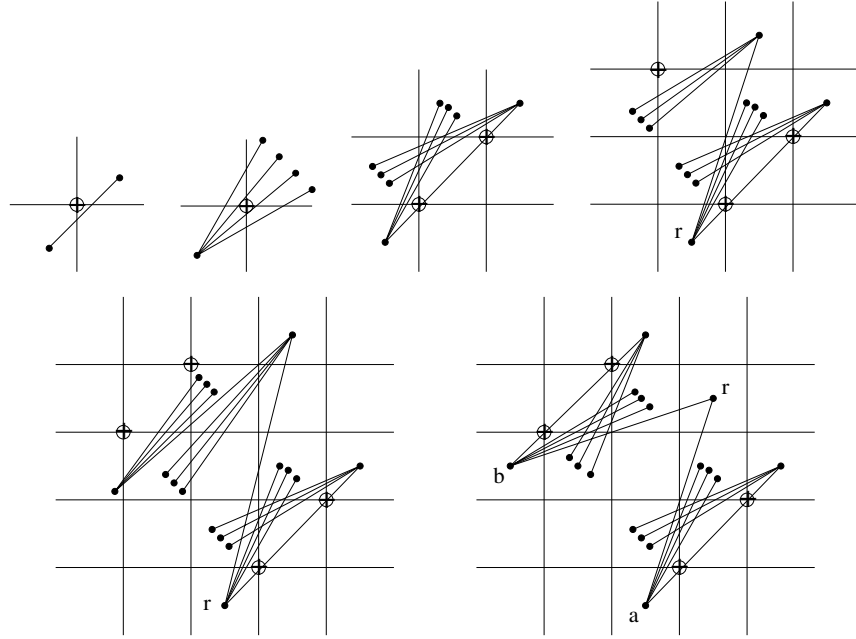


Fig. 2. From left to right, All WRG trees of diameter 1 to 6. Any number of vertices can be added in the cells containing three vertices.

Proof. In Figure 2, all maximal WRG trees are represented. They are ordered by their diameter from left to right. Any number of vertices can be added in the cells where there are at least three vertices.

As one can see in Figure 2 (upper left), all trees of diameter one to three are realizable as WRG.

The unique maximal WRG tree of diameter four (Figure 2 upper right) is obtained by checking all ways to add branches to a path of length four without increasing the diameter and without creating three independent edges. As one can see, branches of length one have been added to the interior vertices, adding a branch of length two would create three independent edges. Adding a branch incident to one of the end-vertices of the path would increase the diameter.

The unique maximal WRG tree of diameter five is constructed similarly (Figure 2 bottom left). If r is the root of the WRG tree, r can have an arbitrary number of children, but at most two of its children can have children (otherwise there would be three independent edges), and at most one of its grandchildren can have children (otherwise the tree would have diameter six).

The unique maximal WRG tree of diameter six is constructed similarly to the case of diameter four (Figure 2 bottom right). If r is the root of the WRG tree, r can have at most two children a and b (otherwise there would be three independent edges), an arbitrary number of grandchildren adjacent to a and b , and a and b can have an arbitrary number of grandchildren all adjacent to exactly one of their children (otherwise there would be three independent edges). \square

Lemma 4 immediately implies the following structural result that is far from a complete characterization, yet narrows substantially the class of graphs representable as WRGs.

Theorem 3. *A witness rectangle graph has at most two non-trivial connected components. If there are exactly two, each has diameter at most three. If there is one, its diameter is at most six.*

Note that the bounds on the diameter are tight: the tree in Figure 2 (right) has diameter six and it is easy to draw the disjoint union of two three-link paths as a WRG, by merging all the vertices in the same cell and removing r from Figure 2 (right), for example.

Corollary 2. *The condition of Theorem 3 is necessary to characterize a WRG but not sufficient. In other words, not all connected graphs with no three independent edges are realizable as a witness rectangle graph.*

Proof. The graph formed from the complete graph K_c by attaching an additional degree-one vertex adjacent to each of the c vertices (see Figure 3), which we call a c -supernova, does not have two independent edges. Nevertheless, by Theorem 6, a

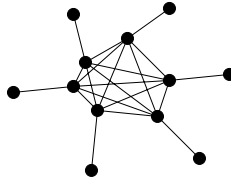


Fig. 3. A six-supernova.

supernova with $c > 4$ is not realizable as a WRG. Indeed, to obtain a dominating set of this graph, one must pick one vertex incident to each of the c edges incident to a leaf. Therefore the domination number is at least c (actually it is c), which implies that graph is not drawable as a WRG, by Theorem 6. \square

In fact, a simple extension of the above argument yields a family of graphs not realizable as WRGs. Start with any graph with at least five vertices and attach five edges to five of its vertices, as above. The resulting graph (which may or may not have three independent edges) has domination number at least five and is therefore not realizable.

3 Two Connected Components

In this section we define a subclass of witness rectangle graphs, called *staircase graphs*. We argue that a WRG with precisely two non-trivial connected components has a very rigid structure. Namely, each of its non-trivial connected components is isomorphic to a staircase graph.

Definition 1. A staircase graph of type IV is a witness rectangle graph, such that the witnesses form an ascending chain (i.e., for every witness, other witnesses lie only in its quadrants I and III) and all the vertices lie above the chain (i.e., quadrant IV of every witness is empty of vertices); refer to Figure 4.

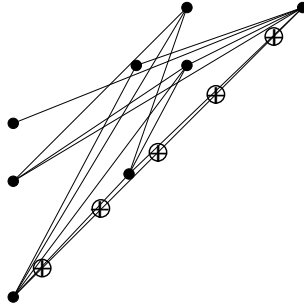


Fig. 4. Staircase graph of type IV.

Staircase graphs of types I, II, and III are defined analogously; they are rotated versions of the above. The type of the staircase graph is determined by which quadrant of all witnesses is empty of vertices.

Note that an induced subgraph of a staircase graph is a staircase graph (of the same type)—a property that immediately follows from the definition and that we will find useful below.

Lemma 5. A combinatorial graph $G = (V, E)$, isomorphic to a staircase graph, is a join $G_1 + G_2$ if and only if it can be realized as a staircase graph of type IV with some witness containing points corresponding to $V(G_1)$ in quadrant I and points corresponding to $V(G_2)$ in quadrant III.

Proof. Suppose $G = G_1 + G_2$ is realizable as a staircase graph. The combinatorial graphs G_1 and G_2 are isomorphic to staircase graphs $\text{RG}^+(P_1, W_1)$ and $\text{RG}^+(P_2, W_2)$ of type IV, which can be obtained, for example, by considering any realization of G as a staircase graph of type IV and dropping the points corresponding to $V(G_2)$ and $V(G_1)$, respectively. Create a staircase graph of type IV isomorphic to G by placing a copy of $\text{RG}^+(P_1, W_1)$ in quadrant I of a new witness w and a copy of $\text{RG}^+(P_2, W_2)$ in its quadrant III.

Conversely, given a staircase graph of type IV isomorphic to G such that some of its vertices (call the set P_1) are in the first quadrant of a witness $w \in W$, and the remaining vertices (call them P_2) are in its third quadrant, it is easily checked that $G = G_1 + G_2$, where G_1 and G_2 are the subgraphs of G induced by (the sets of vertices of G corresponding to) P_1 and P_2 , respectively. \square

Lemma 6. *In a witness rectangle graph, if witness w has no vertices in one of its quadrants, any witness in the empty quadrant is redundant.*

Proof. Suppose quadrant II of w contains a witness w' but no vertices. Hence quadrant II of w' is empty of vertices as well. Suppose w' witnesses edge ab , and a and b are in its quadrants I and III respectively. (They cannot lie in quadrants II and IV, as quadrant II is empty of vertices.) As quadrant IV of w is included in quadrant IV of w' , a and b must be in quadrants I and III of w , respectively, as well. Therefore w witnesses ab . Since this argument applies to all edges witnessed by w' , w' is redundant. \square

Lemma 7. *In a witness rectangle graph, if no witness has a vertex in its quadrant IV, then the graph is a staircase graph of type IV, possibly after removing some redundant witnesses.*

Proof. First remove any redundant witnesses, if present. Now apply Lemma 6, to the possibly smaller, new witness set, to conclude that every remaining witness has its quadrant IV empty of witnesses as well. Therefore the remaining witnesses form an ascending chain and all vertices lie above it, as in the definition of a staircase graph of type IV. \square

Of course, if the empty quadrant in the above lemma is not IV but I, II, or III, we get a staircase graph of the corresponding type.

Theorem 4. *In a witness rectangle graph with two non-trivial connected components, each component is isomorphic to a staircase graph. Conversely, the disjoint union of two graphs representable as staircase graphs is isomorphic to a witness rectangle graph.*

Proof. We distinguish two cases.

All edges have the same slope: Suppose all edges of components C_1, C_2 have the same slope, say positive. Let ab be an edge of C_1 witnessed by w_1 and cd be an edge in C_2 witnessed by w_2 , with $x(a) < x(b)$ and $x(c) < x(d)$. By Lemma 1, w_1 and w_2 are distinct. The vertices a and b are in quadrants III and I of w_1 , respectively. As c and d are not adjacent to a or b , and as cd doesn't share its

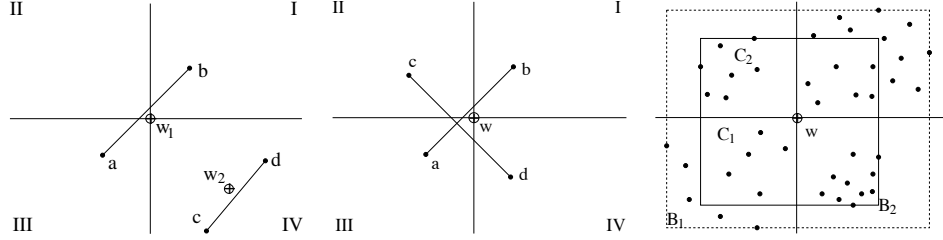


Fig. 5. Cases in the proofs of Lemmas 1, 2, 3, 4 and Theorem 4: Two edges of positive slope (left), two edges of opposite slopes (center). Minimum bounding boxes of $V(C_1)$: B_1 (dashed), and of $V(C_2)$: B_2 (plain). $V(C_1)$ is in quadrants I and III, $V(C_2)$ is in quadrants II and IV (right).

witness with ab , c and d are both in quadrant II or both in quadrant IV of w_1 . Suppose, without loss of generality, that cd is in quadrant IV of w_1 (see Figure 5). By a symmetric argument, ab is in quadrant II of w_2 . This holds for any two edges $ab \in C_1$ and $cd \in C_2$. (Given two edges $ef \in C_1$ and $gh \in C_2$, we say $ef < gh$ if a witness of ef is in quadrant II of a witness of gh , and $gh < ef$ otherwise. We claim that either $ef < gh$ for all choices of edges $ef \in C_1$ and $gh \in C_2$, or $gh < ef$, for all such choices. Otherwise there would have to exist, without loss of generality, a triple of edges $ef, ij \in C_1$ and $gh \in C_2$, with $ef < gh < ij$. This implies that some witnesses w', w'', w''' of ef, gh, ij , respectively, form a descending chain. Considering the relative positioning of the three edges and three witnesses, we conclude that ef and ij must be independent. Hence we have three pairwise independent edges in E^+ , contradicting Lemma 3, thereby proving the claim.) Notice that no vertex of C_1 is in quadrant II of any witness of C_1 or it would be connected to C_2 . Similarly, no vertex of C_2 is in quadrant IV of any witness of C_2 or it would be connected to C_1 . Hence, by Lemma 7, C_1 and C_2 are both staircase graphs.

At least two edges have opposite slopes: There is at least one pair of edges $ab \in C_1$ and $cd \in C_2$ of opposite slopes. Suppose, without loss of generality, that $ab \in E^+$ and $cd \in E^-$. By Lemma 2, ab and cd share a witness w (see Figure 5, center).

Draw isothetic boxes B_1 and B_2 , as follows: B_1 is the minimum bounding box of the vertices of C_1 , while B_2 is its analog for C_2 (see Figure 5, right).

Consider C_1 ; we argue that it is isomorphic to a staircase graph; as we can apply the same reasoning to C_2 , this will imply the first part of the theorem. By Lemma 2, every edge in $E^+ \cap C_1$ shares some witness with cd ; the witness must therefore lie in $B := B_1 \cap B_2$. Let W' be the set of all such witnesses; $w \in W'$. All vertices of C_1 lie in quadrants I and III of every $w' \in W'$, and all vertices of C_2 lie in quadrants II and IV of every w' ; otherwise C_1 and C_2 would be connected. Now remove redundant witnesses from W' , i.e., pick a minimal subset W'' of W such that $H := \text{RG}^+(V(C_1), W')$ coincides with $\text{RG}^+(V(C_1), W'')$. H is a staircase graph and the witnesses of W'' form an ascending chain, by Lemma 7.

Consider the portion of C_1 in quadrant III of w . All witnesses to the left of B_2 and above its lower edge, or below B_2 and to the right of its left edge, have two consecutive quadrants empty of vertices of C_1 (if there were some vertices

of C_1 in these quadrants they would be adjacent to vertices of C_2 defining the borders of B_2). Therefore such a witness would not witness any edges and cannot be present in W .

Hence all remaining witnesses (in quadrant III of w) of edges of C_1 must lie below and to the left of B_2 (and, of course, in B_1). All these witnesses have B_2 in their first quadrant, therefore they witness edges of $E^- \cap E(C_1)$. Let W''' be a minimal subset of such witnesses. Each of them has their quadrant III empty of vertices of C_1 (any vertex of C_1 in quadrant III of such a witness w''' would be adjacent to vertices in C_2 as B_2 is in quadrant I of w'''), therefore by Lemma 7, $\text{RG}^+(V(C_1), W''')$ is a staircase graph.

Consider the lowest leftmost witness w_ℓ of W'' (recall that they form an ascending chain). As shown previously, w_ℓ doesn't have any vertex of C_1 in its second and fourth quadrants. Let V_1 be the set of vertices of C_1 in its first quadrant and let V_2 be the vertices of C_1 in its third quadrant. As shown above, $\text{RG}^+(V_1, W'' \setminus \{w_\ell\})$ and $\text{RG}^+(V_2, W''')$ are staircase graphs, and, therefore, by Lemma 5, $\text{RG}^+(V(C_1), W'' \cup W''')$ is isomorphic to a staircase graph. We apply similar reasoning to quadrant I of B , to conclude that C_1 is a join of at most three staircase graphs and therefore isomorphic to a staircase graph.

C_2 is handled by a symmetric argument (in fact, though it does not affect the reasoning as presented, either C_1 contains negative-slope edges, or C_2 contains positive-slope edges, but not both), concluding the first part of the proof.

Converse: Given two staircase graphs G_1 and G_2 , place a scaled and reflected copy of G_1 in quadrant I of the plane, with witnesses on the line $x + y = 1$ and vertices below the line. Place a scaled and reflected copy of G_2 in quadrant III of the plane, with witnesses on the line $x + y = -1$ and vertices above the line. It is easy to check that the result is a witness rectangle graph isomorphic to $G_1 \cup G_2$. \square

4 What Are Staircase Graphs, Really?

The above discussion is unsatisfactory in that it describes one new class of graphs in terms of another such new class. In this section, we discover that the class of graphs representable as staircase graphs is really a well known family of graphs.

Recall that an *interval graph* is one that can be realized as the intersection graph of a set of intervals on a line, i.e., its set of vertices can be put in one-to-one correspondence with a set of intervals, with two vertices being adjacent if and only if the corresponding intervals intersect. A *co-interval graph* is the complement of an interval graph, i.e., a graph representable as a collection of intervals in which adjacent vertices correspond to disjoint intervals.

Lemma 8. *Graphs representable as staircase graphs are exactly the co-interval graphs.*

Proof. Consider a vertex v in a staircase graph $\text{RG}^+(V, W)$. Without loss of generality, assume the witnesses W lie on the line $\ell: y = x$ and the vertices lie above it. Create an artificial witness on ℓ lying above all vertices. Associate v with the smallest interval I_v of ℓ containing all witnesses lying to the right and below

v as well as the witness immediately above v . It is easily checked that $v \sim v'$ in $\text{RG}^+(V, W)$ if and only if I_v and $I_{v'}$ do not meet. Thus the intersection graph of the intervals $\{I_v \mid v \in V\}$ is isomorphic to the complement of $\text{RG}^+(V, W)$.

Conversely, let H be a co-interval graph on n vertices and $\{I_v\}$ its realization by a set of closed intervals on the line $\ell: x = y$. Extend each interval, if necessary, slightly, to ensure that the $2n$ endpoints of the intervals are distinct. Place $2n - 1$ witnesses along ℓ , separating consecutive endpoints, and transform each interval $I_v = [(a_v, a_v), (b_v, b_v)]$ into point $p_v = (a_v, b_v)$. Let W and P be the set of witnesses and points thus generated. Now I_v misses I_w if and only if $a_v < b_v < a_w < b_w$ or $a_w < b_w < a_v < b_v$, which happens if and only if the rectangle $B(p_v, p_w)$ contains a witness. Hence $\text{RG}^+(P, W)$ is isomorphic to H , as claimed. \square

Theorem 4 and Lemma 8 imply the following more satisfactory statement:

Theorem 5. *The class of graphs representable as witness rectangle graphs with two non-trivial connected components is precisely the class of graphs formed as the disjoint union of zero or more isolated vertices and two co-interval graphs.*

Corollary 3. *Whether or not a given combinatorial graph $G = (V, E)$ with two non-trivial connected components can be drawn as a witness rectangle graph can be tested in time $O(|V| + |E|)$; a drawing, if it exists, can be constructed in the same time.*

Proof. Use the linear-time recognition and reconstruction algorithm for co-interval graphs from [29, 36]. \square

5 Domination number

For a graph G and a subset S of the vertex set $V(G)$, denote by $N_G[S]$ the set of vertices in G which are in S or adjacent to a vertex in S . If $N_G[S] = V(G)$, then S is said to be a *dominating set* of vertices in G . The *domination number* of a graph $G(V, E)$, denoted $\gamma(G)$, is the minimum size of a dominating set of vertices in V [7, 8].

Lemma 9. *The maximum domination number of a connected staircase graph is two.*

Proof. For the lower bound, consider a staircase graph of type IV that is a path of length three. It contains four vertices. It is easy to check that it cannot be dominated by one vertex.

For the upper bound, consider a non-trivial connected staircase graph of type IV (see Figure 6). Let w_i , $1 \leq i \leq k$, be the witnesses in order from left to right, with w_1 being the bottommost leftmost witness and w_k the uppermost rightmost witness. Recall that we assume that the set of witnesses is minimal, hence there is no witness with two adjacent quadrants empty of vertices. This implies that there is at least one vertex d_1 below w_1 and at least one vertex d_2 to the right of w_k . The vertices d_1 and d_2 form a dominating set of size two. Indeed

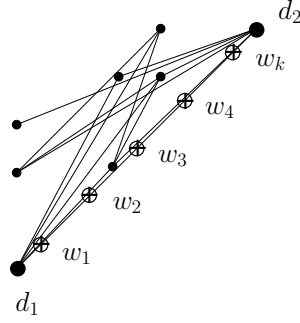


Fig. 6. The two bold vertices form a dominating set for this staircase graph.

all vertices of the graph are in quadrants *I*, *II* and *III* of w_k . The vertex d_1 is adjacent to all vertices in the graph that are in quadrant *I* and *II* of w_k , and d_2 is adjacent to all vertices in quadrant *III* of w_k (since the graph is connected, there are no vertices lying to the left and above of both w_1 and w_k – such vertices would have to be isolated.) \square

Theorem 6. *The maximum domination number of a witness rectangle graph with non-trivial connected components is four.*

Proof. We will consider three cases:

- i) There is a witness w in WRG such that there is at least one vertex in each quadrant of w . Pick one vertex in each quadrant, they form a dominating set of size four. Indeed, the vertices in quadrant *I*, *II*, *III*, and *IV* are respectively adjacent to all vertices in quadrant *III*, *IV*, *I*, and *II*.
- ii) There is a witness w in WRG such that it has two opposite quadrants with at least one vertex in each of them, and two opposite quadrants empty of vertices. Pick one vertex in each of the non-empty quadrants, they form a dominating set of size two. Indeed suppose without loss of generality that the non-empty quadrants are *I* and *III*. Any vertex in quadrant *I* is adjacent to all vertices in quadrant *III*, and any vertex in quadrant *III* is adjacent to all vertices in quadrant *I*.
- iii) All witnesses have one quadrant empty of vertices and three adjacent quadrants containing at least one vertex each. This is the last case as the set of witnesses is minimal which implies that there is no witness with vertices in one quadrant only or in two adjacent quadrants only.

Let the witnesses with one quadrant empty be the witnesses of type *I*, *II*, *III*, and *IV* with the number of the type of witness corresponding to the number of the empty quadrant. For each type of witness, all witnesses of this type witness the edges of a staircase graph of this type, which is a subgraph of the WRG. The union of the four staircase graphs defined by the four types of witnesses is the original WRG (see Figure 7, for clarity the edges are not represented).

By definition of a staircase graph, all its witnesses form a monotone ascending

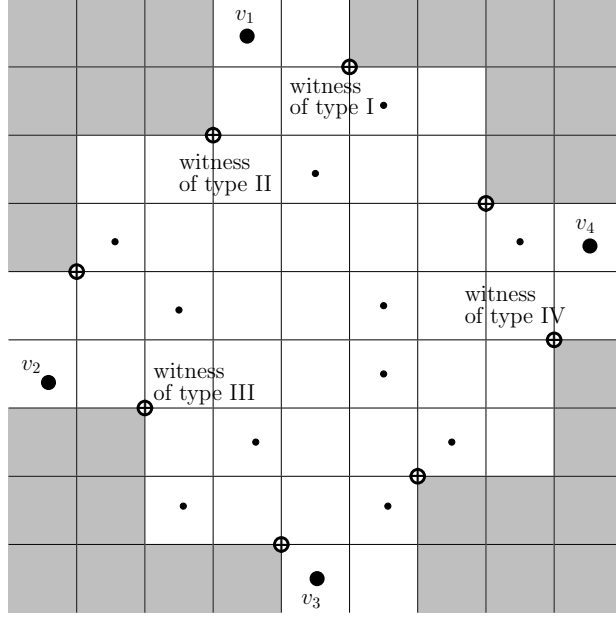


Fig. 7. WRG graph with all witnesses having exactly one quadrant empty. The bold black vertices are the dominating set.

(for staircase graphs of type *II* and *IV*) or descending (for staircase graphs of type *I* and *III*) chain. Moreover notice that all witnesses of type *II* must be in quadrant *II* or *III* of all witnesses of type *I*, all witnesses of type *III* must be in quadrant *III* or *IV* of all witnesses of type *II*, all witnesses of type *IV* must be in quadrant *IV* or *I* of all witnesses of type *III*, and all witnesses of type *I* must be in quadrant *I* or *II* of all witnesses of type *IV*. Reciprocally all witnesses of type *I* must be in quadrant *I* or *IV* of all witnesses of type *II*, all witnesses of type *II* must be in quadrant *I* or *II* of all witnesses of type *III*, all witnesses of type *III* must be in quadrant *II* or *III* of all witnesses of type *IV*, and all witnesses of type *IV* must be in quadrant *III* or *IV* of all witnesses of type *I*. If one of these constraints is violated, one witness would have two adjacent quadrants that are empty, and therefore the set of witnesses would not be minimal, a contradiction.

From this set of constraints it follows that either the vertices in quadrant *II* of the topmost witness of type *I* are included in quadrant *I* of the topmost witness of type *II* and/or the vertices in quadrant *I* of the topmost witness of type *II* are included in quadrant *II* of the topmost witness of type *I*. Hence there is a vertex v_1 in the intersection of quadrant *II* of the topmost witness of type *I* and quadrant *I* of the topmost witness of type *II*. Similarly there is a vertex v_2 in the intersection of quadrant *III* of the leftmost witness of type *II* and quadrant *II* of the leftmost witness of type *III*; there is a vertex v_3 in the intersection of quadrant *IV* of the bottommost witness of type *III* and quadrant *III* of the bottommost witness of type *IV*; and

finally there is a vertex v_4 in the intersection of quadrant I of the rightmost witness of type IV and quadrant III of the rightmost witness of type I . Following the proof of Lemma 9, v_1 and v_2 form a dominating set for the staircase graph of type II , v_2, v_3 form a dominating set for the staircase graph of type III , v_3, v_4 form a dominating set for the staircase graph of type IV , and v_4, v_1 form a dominating set for the staircase graph of type I . Therefore v_1, v_2, v_3, v_4 form a dominating set for WRG.

If there are witnesses of fewer than four types, four vertices are also sufficient. Indeed as shown in the proof of Lemma 9, each staircase graph requires two vertices in its dominating set, and as we have seen above, staircase graphs of consecutive types share a dominating vertex.

□

6 Mutual witness graphs and set separability

In [30], Manabu Ichino and Jack Sklansky introduce “*mutual neighborhood graph*” defined by a pair of point sets (P, W) in the plane. This concept coincides with that of the negative witness rectangle graph $RG^-(P, W)$ in which there is an edge between two points p, q in P if and only if $B(p, q)$ does not contain any witness $w \in W$. For a fixed pair of sets A, B , they focus on the relation between the two negative witness rectangle graphs $RG^-(A, B)$ and $RG^-(B, A)$. In particular, they claim that if $RG^-(A, B)$ and $RG^-(B, A)$ are complete graphs, then the pair (A, B) is linearly separable. Unfortunately this is not true, as demonstrated by the counter-example in Figure 8.

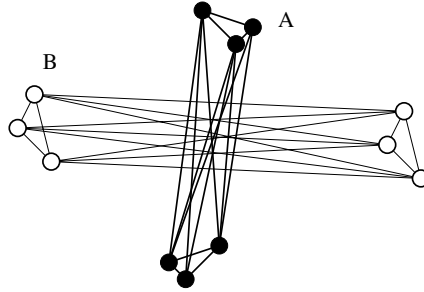


Fig. 8. $RG^-(A, B)$ and $RG^-(B, A)$ are complete (say A is the set of black points and B the set of white points) but the pair (A, B) is not linearly separable.

In [22], the authors have studied some similar properties of corresponding witness proximity graphs in which one set is the set of vertices, the other set is the set of witnesses and inversely, and they prove that for the negative witness Delaunay graph (DG, introduced in [3]), given two disjoint sets of points A and B , if $DG^-(A, B)$ and $DG^-(B, A)$ are complete, the pair (A, B) is circularly separable, but not necessarily linearly separable. On the contrary, for the negative witness Gabriel graph ($GG^-(P, W)$, introduced in [4]), given two disjoint sets of points A and B , if $GG^-(A, B)$ and $GG^-(B, A)$ are complete, the pair (A, B) is linearly separable.

7 Combining positive and negative witnesses

In this section we define a variant of witness rectangle graphs that combine positive and negative witnesses. Given a set of vertices P in the plane, a set W^+ of positive, and a set W^- of negative witnesses, we define $\text{RG}^\pm(P, W^+, W^-)$ as follows. We assume that $P \cup W^+ \cup W^-$ has no two points with the same x -coordinates and no two points with the same y -coordinates. Define $S(p, q) := |B(p, q) \cap W^+| - |B(p, q) \cap W^-|$. The graph has points of P as vertices and $p \sim q$ if and only if $S(p, q) > 0$. In words, p is adjacent to q whenever the rectangle defined by them has more positive witnesses than negative ones.

This class of graphs is a proper generalization of both the positive and negative witness rectangle graphs, for an appropriate choice of the sets W^+ and W^- . In fact, *any* graph can be realized:

Theorem 7. *Any combinatorial graph G on n vertices can be drawn as a witness rectangle graph with positive and negative witnesses using at most $(n-1)^2$ witnesses.*

Proof. Consider an $n \times n$ integer grid. Identify the vertices of G with the points $p_i = (i, i)$ on the diagonal, for $i = 1, \dots, n$, in an arbitrary but fixed order (see Figure 9).

We never put witnesses on the grid lines. We will place witnesses in the (open) squares of the grid, maintaining the following properties: (a) each square contains at most one (positive or negative) witness, (b) for any $p, q \in V(G)$, $S(p, q)$ is either zero or one, and (c) $S(p, q) = 1$ if and only if $p \sim q$.

The process begins with $W^+ = W^- = \emptyset$. We start with the diagonal containing vertices and work our way outward. We maintain the following invariant: property (a) holds all the time, and, at step k properties (b) and (c) hold for all pairs p_i, p_j with $|i - j| \leq k$. When k reaches $n - 1$, we are done.

Base case ($k = 1$): For every pair of consecutive vertices p_i, p_{i+1} , we put a positive witness into $B(p_i, p_{i+1})$ if $p_i \sim p_{i+1}$ and place no witness otherwise.

The invariant is clearly satisfied (see Figure 9 left).

Inductive step ($k > 1$): Consider a pair of vertices p_i, p_{i+k} . Let $X := B((i, i+k-1), (i+1, i+k))$ and $Y := B((i+k-1, i-k), (i+k, i-k+1k))$. We write S_X for $|X \cap W^+| - |X \cap W^-|$ and define S_Y similarly. Since there are no witnesses on the grid lines, by inclusion-exclusion principle we have $S(p_i, p_{i+k}) = S_X + S_Y + S(p_{i+1}, p_{i+k}) + S(p_i, p_{i+k-1}) - S(p_{i+1}, p_{i+k-1})$. By the inductive hypothesis, each of the numbers $S(p_{i+1}, p_k)$, $S(p_i, p_{i+k-1})$, $S(p_{i+1}, p_{i+k-1})$ is either zero or one. In particular, the last three terms sum up to an integer between -1 and 2 . To maintain the invariant, we need $S(p, q)$ to be 0 or 1 , depending on whether $p \sim q$. Hence the combined contribution of X and Y has to be between -2 and $+2$. However, $S_X + S_Y$ can be arranged to have any value from $-2, -1, 0, 1, 2$ by placing one witness of an appropriate sign, or no witness, in X and in Y (see Figure 9 right).

Repeating this process for all pairs (p_i, p_{i+k}) and noticing that the each pair of vertices uses a pair of different boxes (X, Y) , we complete the inductive

step: Every square of the grid still contains at most one witness, and for all pairs of vertices (p_i, p_j) up to k apart, the box $B(p_i, p_j)$ contains an equal number of positive and negative witnesses if $p_i \not\sim p_j$, and positive witnesses outnumber negative ones by one, otherwise. \square

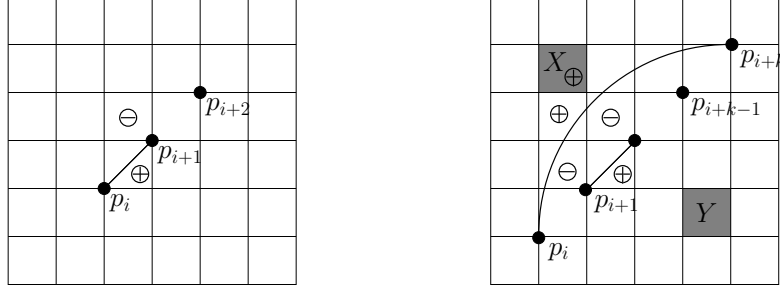


Fig. 9. Vertices are the black disks, positive witnesses are the white disks with a cross, and negative witnesses are the white disks with a minus sign.

The proof suggests a number of open problems: Is it possible that a significantly smaller number of witnesses is sufficient to realize all possible combinatorial graphs? We conjecture that a quadratic number of witnesses is necessary in the worst case. A related open problem is to find a “good” set of vertices to realize a given graph, as it can be proved that every graph can be realized on top of any point set whose elements have no duplicated x or y coordinates, and the number of witnesses depends on the specific initial set. Another open problem is to find classes of graphs that are realizable with much fewer witnesses, say, $O(n)$ or $O(1)$. Also attaching real-number weights to witnesses may possibly allow one to realize graphs with fewer witnesses.

8 How to Stab Rectangles, Thriftily

Let P be a set of n points in the plane, and let S be some given family of geometric regions, each with at least two points from P on its boundary. The problem of how many points are required to stab all the elements of S using a second set W of points has been considered several times for different families of regions [3, 4, 17, 32, 43]. For example, among the shapes previously investigated were the family of triangles with vertices in S and the family of disks whose boundary passes through two points of P . This family of problems has a natural formulation in terms of appropriate witness graphs.

We consider here a variant of this problem that is related to WRGs, in which we focus on the family R of all open isothetic rectangles containing two points of P on their boundary and assume that the points of P have no repeated x - or y -coordinates. We denote by $st_R(n)$ the maximum number of piercing points that are required, when all sets P of n points are considered. Stabbing all the rectangles that have p and q on their boundary is equivalent to just stabbing $B(p, q)$. Therefore we see that

$$st_R(n) = \max_{|P|=n} \min\{|W| : \text{RG}^+(P, W) = K_{|P|}\}.$$

Theorem 8. *The asymptotic value of $st_R(n)$ is $2n - \Theta(\sqrt{n})$.*

Proof. We first construct a set Q of n points, no two of them with equal abscissa or ordinate, that admits a set of $2n - \Theta(\sqrt{n})$ pairwise openly-disjoint rectangles, whose boundary contains two points from Q , which will imply the lower bound.

Start with a grid of size $\sqrt{n} \times \sqrt{n}$, then rotate the whole grid infinitesimally clockwise, and finally perturb the points very slightly, so that no point coordinate is repeated and there are no collinearities. The desired set of rectangles contains $B(p, q)$ for every pair of points $p, q \in Q$ that were neighbors in the original grid; refer to Figure 10.

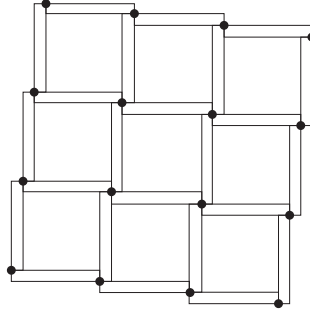


Fig. 10. Construction of a set Q with a large number of rectangles with disjoint interiors, each one with two points from Q on its boundary; every point of Q participates in four rectangles, with the exception of those on the boundary of the configuration.

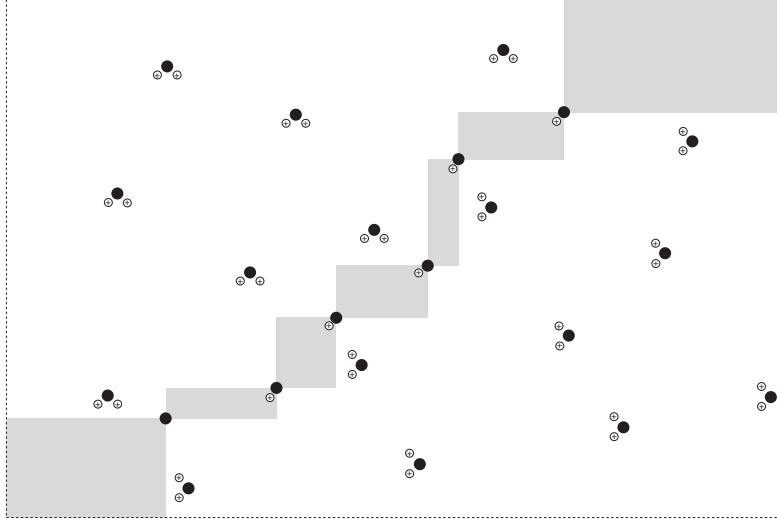


Fig. 11. Construction, for a point set P , of a set W of positive witnesses such that $RG^+(P, W) = K_{|P|}$.

For the proof that $2n - \Theta(\sqrt{n})$ points suffice to stab all the rectangles refer to Theorem 6 in [3]. A sketch of the proof is illustrated in Figure 11. The points

that get only one witness form an ascending chain. A descending chain would be used similarly and one of these alternative chains must have $\Omega(\sqrt{n})$ size. \square

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